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Mechanisms of decay of laminar and turbulent vortices

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The dynamics of an infinitely long one-dimensional vortex and a swirl are compared with the dynamics of a semi-infinitely long trailing vortex and trailing swirl. With increasing distance, the change in the axial velocity difference between the core of the trailing vortex and the surrounding region causes radial convection and some associated axial convection of angular momentum. In laminar or turbulent trailing vortices, we show that under most conditions of interest this is the dominant mechanism for the decrease in the velocities of swirl in the core and corresponding growth of the core. On the basis of theoretical considerations and experimental observations, we show that the axial velocity difference between the core of the trailing vortex and the surrounding region is necessary for the sustenance of turbulence in the vortex core. A theory of the turbulent trailing vortex is developed on the basis of these mechanisms and the results are compared with our experimental observations.

1. Introduction

There are a variety of theories and views about unconfined and semi-infinitely or infinitely long turbulent swirls and vortices. These theories are neither confirmed nor refuted by experimental investigations, since very few experiments exist where the effects of initial conditions and extraneous influences have been minimized. We critically examine the assumptions underlying these theories and find them less viable than those advanced in this paper. The results of analysis based on physical processes proposed here are compared with the most recently available experiments.

There are four distinct types of flow which are relevant here, namely a *line swirl*, *line vortex*, *trailing swirl* and *trailing vortex*. These flows together with their velocity, circulation and vorticity profiles are sketched in figure 1. The definitions given below prescribe some properties of velocity, circulation and vorticity profiles for each of the flows.

A *line swirl* and *line vortex* are time-dependent infinitely long flows while a *trailing swirl* and *trailing vortex* are semi-infinitely long steady flows.

Initially, in all these flows vorticity is assumed to be confined to a region of small radius and the moments of all orders of any vorticity component with respect to the z axis exist. The cores of these flows are regions where most of the vorticity is located. This paper is concerned with the growth of these cores or the spread of initially concentrated vorticity owing to viscous and turbulent processes.

A *line swirl* is an infinitely long flow which has a single velocity component $u_\theta(r,t)$, where r is the radial distance from the swirl axis z, the subscript θ refers to the angular

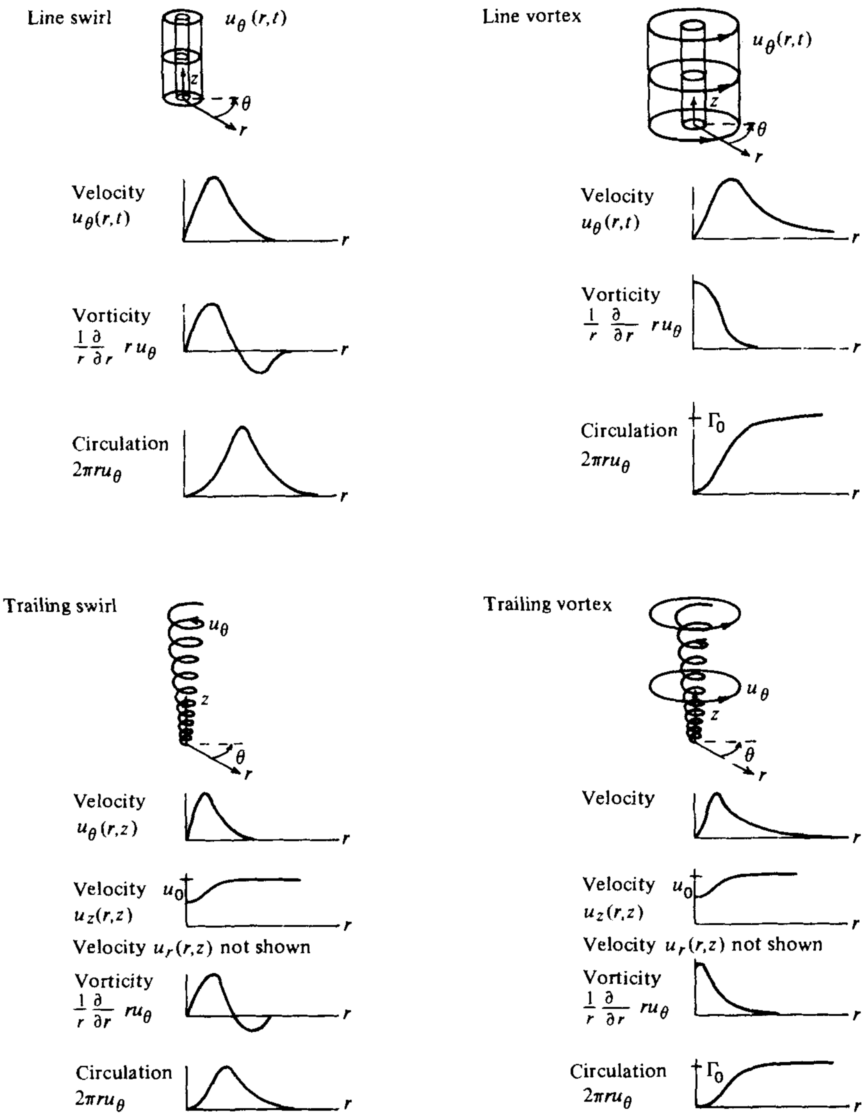

FIGURE 1. Sketch of the four flows and their profiles.

co-ordinate, and t is the time. Its angular momentum ρM per unit axial distance is constant and independent of time, where

$$M = 2\pi \int_0^\infty u_\theta(r,t)\, r^2\, dr \tag{1}$$

and ρ is the density of the fluid. A prime, as in u_θ', will be used to denote the fluctuating part of a quantity. Otherwise the symbol represents either a quantity in a laminar flow or its mean value in a turbulent flow.

The second flow is an infinitely long *line vortex* with one velocity component $u_\theta(r,t)$ such that $u_\theta = \Gamma_0/2\pi r$ for large r with zero velocity at $r = 0$. Its angular momentum per unit distance along the swirl axis is infinite and its rate of change of angular momentum is finite and depends on the viscosity. It is further assumed that $\partial\Gamma^2/\partial r > 0$, where $\Gamma = 2\pi r u_\theta$.

The third flow is a *trailing swirl* with three velocity components $u_z(r, z)$, $u_\theta(r, z)$ and $u_r(r, z)$, where z is the distance along the axis of the trailing swirl measured from its origin. It may be produced by rotating vanes with the axis of rotation parallel to the prevailing uniform steady flow u_0. The vanes may add to or absorb some axial momentum of the prevailing flow. Thus we have a swirl which may have a coaxial jet or a wake. The flux of angular momentum

$$A = 2\pi\rho \int_0^\infty u_z(r, z)\, u_\theta(r, z)\, r^2 dr \tag{2}$$

is independent of z. We assume that, as $r \to \infty$, $u_r = 0$, $u_z = u_0$, and u_θ is bounded and approaches zero rapidly enough for the above integral to exist. See the fourth paragraph from the beginning of this section.

The fourth flow is a *trailing vortex* produced at the tip of a semi-infinite lifting wing in the presence of a prevailing mean flow. The trailing vortex bears some similarity to a trailing swirl in having three velocity components, but with the important difference that, as $r \to \infty$, $u_r = 0$, $u_\theta = \Gamma_0/2\pi r$ and $u_z = u_0$, where Γ_0 is the total circulation. The flux of angular momentum is infinite, but its rate of change

$$dA/dz = 2\pi\rho \int_0^\infty \frac{\partial}{\partial z}(u_z u_\theta)\, r^2 dr \tag{3}$$

is finite and depends on z. It is further assumed that $\partial\Gamma^2/\partial r > 0$.

In analysing the third and fourth flows, all previous investigators have invariably assumed that u_r may be neglected, $u_z = u_0$ and z is replaced by tu_0, thus reducing the trailing swirl to a line swirl and the trailing vortex to a line vortex. We indicate below and subsequently show in detail that this approximation is invalid.

As a trailing swirl or a trailing vortex develops, the swirl velocity u_θ decreases with increasing downstream distance z. Since the pressure at large r is constant, near the axis of a swirl or a vortex this leads to $\partial p/\partial z > 0$, where p is the pressure. This implies divergence of the cores of these flows and an axial velocity difference between the cores and the surrounding regions, which has three important effects on the dynamics of these trailing flows which are absent in a line swirl and a line vortex.

(i) Linear and nonlinear stability analyses (Uberoi, Chow & Narain 1972; Narain & Uberoi 1973) show that a difference in axial velocity between the core and the surroundings destabilizes swirling flows which otherwise would be stable.

(ii) There may be significant and sometimes dominant radial and associated axial convection of angular momentum.

(iii) The range of downstream distances over which dynamic self-similarity exists may be limited.

The importance of these effects decreases with decreasing rate of spread of these flows. However, in the study of trailing flows with swirl the emphasis is on their rates of growth rather than on the final stages where they have practically ceased to grow.

2. Laminar line swirl, line vortex and their stabilities

It is important to consider the dynamics and stability of the basic laminar flow, a part of which may become turbulent. In swirling flows the basic laminar flow may enhance, diminish or even quench turbulence in its interior. These, sometimes strong, stabilizing or destabilizing effects must be considered when postulating turbulent stresses in these flows.

The equation of motion for a line swirl or a line vortex is

$$\frac{\partial}{\partial t} r^2 u_\theta = \frac{\partial}{\partial r}\left(\frac{r^2 \tau}{\rho}\right) = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right), \tag{4}$$

where the viscous stress $\tau = \rho\nu r\,\partial(u_\theta r^{-1})/\partial r$ and ν is the kinematic viscosity.

A swirl of finite M diffuses out owing to viscosity and shares its angular momentum with the surrounding fluid, which is set into motion. Its Reynolds number is $(M/t)^{\frac{1}{2}}/\nu$. A known self-similar swirl is

$$\Gamma = 2\pi r u_\theta = \frac{M}{2\nu t}\left(\frac{r^2}{4\nu t}\right) \exp\left(-r^2/4\nu t\right). \tag{5}$$

In a line vortex $u_\theta \simeq \Gamma_0/2\pi r$ for $r \to \infty$ and the rate of change of angular momentum is determined from (4); thus

$$2\pi \int_0^\infty \frac{\partial}{\partial t} r^2 u_\theta \, dr = [r^2\tau/\rho]_0^\infty = -2\nu\Gamma_0. \tag{6}$$

The flow has only one non-zero velocity component $u_\theta(r,t)$ and hence there is no convection of angular momentum. The above rate of change of angular momentum of the entire line-vortex flow consisting of the core and the nearly potential surrounding flow is independent of the detailed distribution of the vorticity or u_θ in the interior of the vortex. It depends only on the fact that $u_\theta \simeq \Gamma_0/2\pi r$ for $r \to \infty$. In other words, this rate depends only on the viscous torque at $r \to \infty$. If we assume a finite ν then there are stresses but no net force on a fluid element in the potential flow surrounding the core where most of the vorticity resides. The angular momentum is lost from the interior of the vortex through the potential-flow region to the region $r \to \infty$. The core over which most, say 95 %, of the total vorticity or Γ_0 is distributed grows and so does its angular momentum. This is due to infinite angular momentum surrounding any finite though growing interior region. The Reynolds number is Γ_0/ν.

A known self-similar solution for the line vortex is

$$\Gamma = 2\pi r u_\theta = \Gamma_0[1 - \exp\left(-r^2/4\nu t\right)]. \tag{7}$$

We may combine (5) and (7) to get a vortex–swirl combination

$$\Gamma = \Gamma_0 \left[1 + \left\{\frac{M}{2\nu t \Gamma_0}\left(\frac{r^2}{4\nu t}\right) - 1\right\} \exp\left(-r^2/4\nu t\right)\right]. \tag{8}$$

Equation (7) shows that the circulation in a line vortex increases monotonically with radius, reaching a constant value Γ_0. The total vorticity in a swirl is zero and Γ increases and then decreases to zero for large r. In the example of a line vortex–swirl given by (8), Γ overshoots Γ_0, then decreases to Γ_0 for $r \to \infty$. This overshoot decreases and becomes relatively insignificant with time and may be considered as a decaying 'initial' disturbance.

This is a special case of a general result. If the initial vorticity is axisymmetric and is distributed over a finite area around the origin then its subsequent distribution can be found from (4). After a long time the vorticity and velocity distributions will have forms corresponding to a total vorticity originally at the origin, any initial vorticity distribution of opposite sign and zero total value having no significant influence. If, however, the total vorticity is zero, then the initial distribution of the vorticity determines the subsequent state of the vorticity and the velocity.

We may look at the situation from the point of view of dynamics. The angular momentum associated with a finite total vorticity is infinite but the angular momentum is finite if the distribution of vorticity is such that the total vorticity or Γ_0 is zero. As time progresses the former will dominate the latter. We are assuming that the vorticity is initially concentrated near the axis.

These well-known results are presented to contrast some properties of laminar flows with corresponding properties of the flows with the same overall parameters but in a turbulent state. For example, in a laminar line vortex–swirl the dynamics of any region of overshoot where $\Gamma > \Gamma_0$ become in time unimportant to the dynamics of the main vortex, while in a turbulent line vortex–swirl Govindaraju & Saffman (1971) assert that at high Reynolds numbers the overshoot ($\Gamma > \Gamma_0$) is the main growth mechanism of the turbulent vortex.

The virtual origins of the time t for a line swirl and line vortex may not be the same. This becomes insignificant as t becomes large. However, for both small and large t we define a line vortex–swirl as a flow which behaves like a vortex for sufficiently large r and for which $d\Gamma^2/dr < 0$ for some finite range of r. This slight generalization is necessary for the purpose at hand, which is to study the stability and turbulence in such flows.

An important criterion, based on analysis assuming inviscid flow, for the stability of swirling flows with *one* velocity component $u_\theta(r,t)$ is (Rayleigh 1916; Chandrasekhar 1961, p. 284)

$$d\Gamma^2/dr > 0. \tag{9}$$

A line vortex is stable at all times. A swirl is unstable. A line vortex–swirl is unstable and its interior may become turbulent at a sufficiently high Reynolds number. However, as time progresses the flow will tend to stabilize and production of turbulence already created will decay owing to viscosity. In practice, flows approximating a line vortex are used to stabilize unstable fluid configurations such as a vortex-stabilized electric arc (Chow & Uberoi 1972). Experiments lend support to the above criterion (9) without regard to any limitations of the stability theories cited.

In the case in which there is an initial thin shear layer such that u_θ changes rapidly or discontinuously, instabilities and turbulence may develop in this unstable shear layer. However, in time the shear layer will be smoothed out. The turbulence will be in the form of an initial disturbance and will not be sustained just as for the case $\partial\Gamma^2/\partial r < 0$ discussed above.

We have made some simple observations to check the stability considerations. Water was injected tangentially all along the inner wall of a transparent vertical cylindrical vessel 30 cm in diameter and 50 cm long which had a central drain in its flat bottom and was nearly full of water. After the 'bathtub' vortex had been set up, the fluid in the centre was made turbulent by stirring it randomly or by spinning a 0·6 cm diameter rod spanning the entire length of the cylinder in a direction the same as or opposite to that of the main vortex. The turbulent flow was made visible by painting the stirrer or the rod with water-soluble ink. Care was taken to limit the disturbance to a short time period, so as to confine the initial disturbance to a cylindrical region about 3 cm in diameter, which may be considered small in size compared with the main vortex. The Reynolds number Γ_0/ν of the vortex was about 10^3. The drainage was reduced after each repetition of the experiment just described, there being no drainage in the last repetition of the experiment. The reduction of the drainage was intended to approximate better a flow with one velocity component $u_\theta(r,t)$ by a 'bathtub' vortex. In every case the initial turbulence decayed and was not sustained at the expense of the energy of the relatively slowly changing main motion of the vortex.

In another experiment, at $\Gamma_0/\nu = 7{\cdot}8 \times 10^4$, we have made detailed velocity measurements in the trailing vortex of a wing of laminar-flow airfoil. Near the wing tip and in the vortex core u_z exceeded u_0, the prevailing velocity, and the core flow was turbulent. Some distance downstream $u_z \simeq u_0$ and hence the trailing vortex flow was almost a line vortex: no substantial axial velocity differences existed between the core and the surrounding fluid. The existing turbulence in the core mostly disappeared. Further downstream $u_z < u_0$ and laminar unstable modes appeared which were due to instabilities caused by the velocity deficit which now appeared (Singh & Uberoi 1976). This further substantiates our claim that turbulence cannot be sustained in a line vortex (i.e. without a difference between the axial velocities of the core and the surrounding fluid) no matter how large the Reynolds number.

3. Turbulent line swirl, line vortex and their combination

In § 1 we have defined the four flows and in § 2 we have considered the dynamics and instabilities of two of these flows, namely a line swirl and line vortex. We now consider the possible turbulent state of these two flows.

A line swirl is unstable according to the criterion (9) based on the assumption of an inviscid fluid. Thus at sufficiently high Reynolds numbers it should become unstable, which is consistent with experience. One can use hypotheses about the turbulent shear in a swirl which are standard in theories of turbulent free shear flows to develop the theory for the turbulent line swirl (Uberoi 1977*a*).

A line vortex is stable unless it has a swirl superimposed on it; i.e. there is a finite radial region where $d\Gamma^2/dr < 0$. If a swirl is superimposed on a line vortex, turbulence will develop in the form of an initial disturbance and decay at a faster rate than the asymptotic rate of growth of the vortex core. Thus, in contrast to a line swirl, we cannot use standard shear-flow hypotheses for a line vortex or a line swirl–vortex combination. However, there are several theories of sustained turbulence in a line vortex. We show below that the results obtained by other investigators who applied standard assumptions for free turbulent shear flow to the turbulent line vortex are erroneous. They assumed that there exists three-dimensional turbulence. However there is only one mean velocity component $u_\theta(r,t)$. The equation governing a turbulent line swirl, a turbulent line vortex (assuming that it exists) or their combination is

$$\frac{\partial}{\partial t}(r^2 u_\theta) = \frac{\partial}{\partial r} r^2 \left(-\overline{u_r' u_\theta'} + \nu r \frac{\partial}{\partial r}\frac{u_\theta}{r}\right), \tag{10}$$

where $-\rho\overline{u_r' u_\theta'}$ is the turbulent stress, which is confined to a finite central core and vanishes outside it. This is consistent with the fact that for all known shear flows without constraining boundaries the turbulence is always confined to a finite domain across the flow and is separated from the non-turbulent flow by an irregular sharp boundary. Turbulence has never been known to diffuse out to infinity across the free shear flows.

Consider the total rate of change of angular momentum of a turbulent line swirl–vortex combination or a line vortex (assuming that it exists):

$$2\pi\int_0^\infty \frac{\partial}{\partial t}(r^2 u_\theta)\,dr = -2\nu\Gamma_0, \tag{11}$$

where we have assumed that $u_\theta \simeq \Gamma_0/2\pi r$ for $r\to\infty$ and that the turbulent shear is confined to a finite core and does not affect the ultimate transfer of angular momentum through the potential flow surrounding the vortex core. See also the discussion preceding (6).

Squire (1965) was the first to consider a vortex with a turbulent core. In effect he assumed that 'turbulent' kinematic viscosity is

$$\nu_t = \alpha\Gamma_0, \tag{12}$$

where α is a constant. The solution is given by (7) with ν replaced by $\alpha\Gamma_0$. Since the turbulence is confined to a finite radius, ν_t should vanish as $r\to\infty$. Squire's assumption allows far too much angular momentum to escape to infinity; the correct amount is determined by (11).

We may try to save Squire's solution by stipulating that it is valid only for a vortex in the presence of uniform atmospheric turbulence, where a constant ν_t may be used. However there are serious difficulties with this artifice. A constant turbulent kinematic viscosity due to atmospheric turbulence has nothing to do with Γ_0, which is associated with the vortex. Let the atmospheric turbulence be strong enough to interact with the vortex. In the potential part of the vortex, owing to the spatially varying rate of strain, the interaction would vary spatially and with time. Consequently ν_t cannot be assumed constant. Further, we cannot assume that the entire vortex interacts significantly with the atmospheric turbulence while the outer flow is still potential with $u_\theta = \Gamma_0/2\pi r$ for all time.

Hoffman & Joubert (1963) considered radial transfer of angular momentum in the turbulent core of a line vortex. Using certain assumptions, they concluded that the circulation varies logarithmically with radial distance in the region of maximum u_θ. They failed to show how the momentum is transferred to large radial distances; this transfer is determined by viscosity and is given by (11).

The total rate of change of angular momentum is due to viscosity and cannot exceed that given by (11). If we *insist* on growth of the core, i.e. a decrease of its swirl velocities faster than that caused by viscosity, then the outer flow *must* speed up, since the total angular momentum must be conserved except for a small loss due to viscosity. It follows that the circulation in the region of potential flow where the flow speeds up must exceed or overshoot Γ_0. Various elaborate theories have been developed to 'prove' the existence of a circulation overshoot. In fact, the overshoot is a direct consequence of the *insistence* mentioned above, which may take many different forms.

Govindaraju & Saffmann (1971) assume that $r^2\overline{u_r' u_\theta'}$ and ru_θ are functions of $r/t^{\frac{1}{2}}$, in which case (10) becomes an ordinary differential equation. The insistence is contained in this functional dependence; i.e. the maximum value of u_θ must decrease and the core size must increase faster than they would owing to viscosity.

Macagno & Macagno (1975) assume that if ϵ is the mean rate of strain then the turbulent kinematic viscosity is

$$\nu_t = \alpha + \tfrac{1}{2}\beta|\epsilon|, \tag{13}$$

where α and β may vary with space and time but are taken to be constant in their analysis. The quantity α here is not related to that in (12). Equation (13) is supposed to include vortex-generated and atmospheric turbulence. In accordance with the above discussion of Squire's work α must equal ν. Equation (13) allows turbulent stresses and hence production of turbulence in the outer potential flow, where ϵ is finite.

The theories discussed here and other such theories were based on the unjustifiable belief that a line vortex is equivalent to a trailing vortex, where t is replaced by z/u_0. The results of such analyses of line vortices were compared with experimental observations of trailing vortices.

Measured data on trailing vortices have been fitted to Squire's solution for a line vortex, although the value of the constant α varies from case to case (Rose & Dee 1963; McCormick, Tangler & Schemel 1968). Owen (1970) has given an explanation for the variations of α with Γ_0/ν. This does not remove the fundamental objections raised above.

4. Laminar trailing swirl, trailing vortex and their instabilities

The equation governing u_θ in these flows is

$$\frac{\partial}{\partial z}(u_0+u_z)\,u_\theta r^2 + \frac{\partial}{\partial r}(u_r u_\theta r^2) = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right), \tag{14}$$

where u_0 is the constant prevailing velocity along the swirl axis and now u_z is the deviation of the axial velocity from u_0. In the core of these flows the swirl velocity u_θ decreases with increasing downstream distance z and therefore $\partial p/\partial z > 0$. The flow in the core diverges, causing significant radial and some associated axial convection of angular momentum. We illustrate this by examining the total rate of change of angular momentum in a trailing vortex. Integrating (14), we have

$$2\pi u_0 \int_0^\infty \frac{\partial}{\partial z} u_\theta r^2\,dr = -2\pi \underbrace{\int_0^\infty \frac{\partial}{\partial z} u_z u_\theta r^2\,dr - [u_r r]_{r\to\infty}\Gamma_0}_{\text{convection}} - \underbrace{2\nu\Gamma_0}_{\text{diffusion}}. \tag{15}$$

In deriving (15) we have assumed that $u_r r = r^3\,\partial(u_\theta r^{-1})/\partial r = 0$ at $r = 0$ and $u_\theta \simeq \Gamma_0/2\pi r$ as $r\to\infty$, so that $2\pi\nu r^3\,\partial(u_\theta r^{-1})/\partial r \to -2\nu\Gamma_0$ as $r\to\infty$.

From the continuity equation, we have

$$[u_r r]_{r\to\infty} = -\int_0^\infty \frac{\partial u_z}{\partial z} r\,dr. \tag{16}$$

Using (16) in (15), we get

$$2\pi u_0 \int_0^\infty \frac{\partial}{\partial z} u_\theta r^2\,dr = \underbrace{\int_0^\infty \frac{\partial}{\partial z}(\Gamma_0-\Gamma)\,u_z r\,dr}_{\text{convection}} - \underbrace{2\nu\Gamma_0}_{\text{diffusion}}. \tag{17}$$

We have assumed that, as $r\to\infty$, $u_z\to 0$ rapidly enough for the integrals in (15)–(17) to exist. However, we have not assumed that $u_z \ll u_0$. Batchelor (1964) has calculated axial flow in a trailing vortex, neglecting radial flow. He neglected terms involving u_r and u_z compared with that involving u_0 in (14), which becomes

$$u_0 \frac{\partial}{\partial z} u_\theta r^2 = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right), \tag{18}$$

which replaces a trailing vortex with a line vortex. The velocity u_θ is given by (7) with t replaced by z/u_0. Using this u_θ, the pressure is calculated from the approximate equation

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r}, \tag{19}$$

where

$$2\pi u_\theta r = \Gamma_0(1-e^{-\xi}) \tag{20}$$

and

$$\xi = u_0 r^2/4\nu z. \tag{21}$$

This leads to

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \left(\frac{\Gamma_0}{2\pi}\right)^2 \frac{u_0}{8\nu z^2}(P\xi)', \tag{22}$$

where

$$P(\xi) = \int_\xi^\infty \frac{(1-e^{-t})^2}{t^2}\, d\xi. \tag{23}$$

The axial velocity u_z is calculated from the above pressure gradient and the equation

$$\rho u_0 \frac{\partial}{\partial z} u_z = -\frac{\partial p}{\partial z} + \rho\nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) u_z. \tag{24}$$

Batchelor's solution for u_z has been criticized for non-uniqueness by Tam (1973), whose argument has in turn been criticized by Herron (1974). We wish to avoid these controversies here since our main interest is to examine the range of validity of the approximations used in deriving the above equations rather than solving them. These difficulties have been examined by us and their resolution is given elsewhere (Uberoi 1978; Uberoi, Shivamoggi & Chen 1979).

We make a few remarks which are relevant to our purpose here. The diffusion equation (24), together with the distribution of sources given by (22), has a unique solution provided that we specify the initial $u_z(r, z_0)$. No one has noticed that as a result of the approximations and simplifications used the integral with respect to z and r of the sources given by (22) is infinite owing to a singularity at $z = 0$. Hence initial conditions cannot be specified at $z_0 = 0$, and no solution exists which is independent of z_0.

Batchelor's formulation neglects the convection of angular momentum in (18). Here our aim is to estimate this neglect relative to diffusion, which is retained; see (15) and (17). This can be done from the governing equations without explicitly solving for u_z.

Let us assume that at z_0

$$u_z(r, z_0) = 0. \tag{25}$$

Owing to the pressure gradient u_z will, of course, change. This is a reasonable initial condition since our main interest here is the change in u_z due to the prescribed pressure gradient.

The radial convection and associated axial convection of angular momentum at z_0 are given by (17); thus

$$\int_0^\infty \left[\frac{\partial}{\partial z} u_z(\Gamma-\Gamma_0)\right]_0 r\,dr = \int_0^\infty \left[(\Gamma-\Gamma_0)\frac{\partial u_z}{\partial z}\right] r\,dr$$
$$= \frac{\Gamma_0^3}{16\pi^2 z_0 u_0}\int_0^\infty e^{-\eta}(P\eta)'d\eta \simeq \frac{\Gamma_0^3}{16\pi^2 z_0 u_0}\times\frac{1}{2}, \tag{26}$$

where we have made use of (20)–(25) and the value of the definite integral is approximately $\frac{1}{2}$. Using (26) and (17), the ratio is given by the expression

$$\text{convection/diffusion} \simeq (\Gamma_0/\nu)(\Gamma_0/64\pi z_0 u_0) \simeq (\Gamma_0/\nu)(c/400 z_0), \tag{27}$$

where we have assumed that the trailing vortex is generated by a semi-infinite wing of chord c and that $\Gamma_0 = \frac{1}{2}cu_0$. Since the vortex is essentially a high Reynolds number phenomenon, $\Gamma_0/\nu \gg 1$ and z_0/c must be large to make this ratio much smaller than unity. In general, radial and associated axial convection of angular momentum cannot be neglected.

The above procedure allows us to estimate the neglected convection of angular momentum only at z_0 and for the specific initial condition given by (25). However, at any station z we can calculate a significant part, namely the radial flow of angular momentum $\Gamma_0(u_r r)_{r\to\infty}$, of the neglected convection by using the continuity equation and integrating (24) with respect to r; see (15)–(17). The ratio of radial flux of angular momentum to the diffusion of angular momentum at any station z is twice that given by (26). Therefore (26) may be taken as a reasonable estimate of the terms neglected in (18) at any station by replacing z_0 by z.

Moore & Saffman (1973) have calculated the axial velocity in the core of a vortex for which at $z = 0$ and $u_\theta = \beta r^{-n}$, where β is a constant and $0 < n < 1$. They also neglected radial and associated axial convection of angular momentum. Using their axial velocity, we find that the requirement for neglecting these is that

$$\beta^2/u_0^2\left(\frac{\nu z}{u_0}\right)^n \ll 1. \tag{28}$$

They were concerned with the axial velocity during vortex-sheet roll-up near the wing or at small z, where the flow is essentially three-dimensional and u_r cannot be neglected. The condition (28) may be satisfied at large z, but then the vortex sheet is rolled up, which corresponds to the spreading of the vorticity in the core of a line vortex, which we are considering here.

The present discussion shows that, for those distances from the origin of interest where the vortex is changing significantly, the dominant mechanism for a decrease in the swirl velocities in the core of a laminar trailing vortex is radial and associated axial convection of angular momentum.

Another important effect of axial flow or a difference in u_z between the core and the surroundings is that a flow becomes unstable which was otherwise stable according to criterion (9) (Uberoi *et al.* 1972).

A trailing vortex is further destabilized when a trailing swirl is added to it such that there is a finite radial distance r for which $d\Gamma^2/dr < 0$, in the same manner as for a line vortex.

5. Turbulent trailing vortex

On dimensional grounds we may write the functional dependence of the circulation $\Gamma\,(=2\pi r u_\theta)$ as

$$\Gamma/\Gamma_0 = \gamma(ru_0/\Gamma_0, zu_0/\Gamma_0;\ \Gamma_0/\nu). \tag{29}$$

The equation governing u_θ under the approximation of a slender vortex core is

$$\frac{\partial}{\partial z}(u_0+u_z)\,u_\theta r^2 = \frac{\partial}{\partial r}\,r^2\left(-u_r u_\theta - \overline{u_r' u_\theta'} + \nu r\frac{\partial}{\partial r}\frac{u_\theta}{r}\right). \tag{30}$$

Integrating this equation, we have

$$u_0\int_0^\infty \frac{\partial}{\partial z}\Gamma r\,dr = -\int_0^\infty \frac{\partial}{\partial z}\Gamma r\,dr - (u_r r)_{r\to\infty}\Gamma_0 - 2\nu\Gamma_0 \tag{31}$$

$$= -\underbrace{\int_0^\infty \frac{\partial}{\partial z}u_z(\Gamma-\Gamma_0)\,r\,dr}_{\text{convection}} - \underbrace{2\nu\Gamma_0}_{\text{diffusion}}, \tag{32}$$

where we have assumed that turbulence in the trailing vortex vanishes as $r\to\infty$ and have made use of the continuity equation. See also the discussion preceding (16) and (17). It is well known that in all flows without constraining boundaries the turbulent fluid is separated from non-turbulent fluid by a sharp irregular boundary. Velocity fluctuations decay very rapidly as we move from turbulent into non-turbulent fluid. Further, measurements (Singh 1974; Uberoi 1974) in a trailing vortex show that $r(\overline{u_\theta'^2})^{\frac{1}{2}}$ and $r(\overline{u_r'^2})^{\frac{1}{2}}$ both vanish as $r\to\infty$. Hence $r^2\overline{u_r' u_\theta'}\to 0$ as $r\to\infty$. We are concerned here with the spread of turbulence which is initially concentrated near the axis of the trailing vortex. In all turbulent flows experimentally investigated thus far, the irregular front separating the turbulent from the non-turbulent fluid propagates at a finite rate rather than diffusing to infinity. The vanishing of $r^2\overline{u_r' u_\theta'}$ is consistent with all known experimental facts about the trailing vortex and turbulent flows in general. Assuming that $r^2\overline{u_r' u_\theta'}$ is finite as $r\to\infty$ would lead us to the same difficulty as in Squire's work (1965). Angular momentum far in excess of that allowed by laminar viscosity of the fluid would escape to infinity owing to turbulent stresses.

It follows from (32) that the radial and associated axial convection of angular momentum are important if the turbulent vortex grows faster than the laminar vortex and there is no overshoot, or $\Gamma \leqslant \Gamma_0$. The experiments determine quantitatively the importance of convection relative to diffusion.

The velocities u_θ and u_z in a turbulent trailing vortex behind an airfoil have been measured by Singh (1974) and Uberoi (1974) at $\Gamma_0/\nu = 2{\cdot}1\times10^4$. Unfortunately, the convection of angular momentum cannot be accurately calculated from the measured u_z. We have calculated the first and the last terms in (32) using the measured u_θ and thus determined that diffusion is about 1 % of the convection of angular momentum. Therefore the dynamics of a turbulent trailing vortex are independent of Γ_0/ν at least at the Reynolds number of the experiment, and for the range of $u_0 z/\nu$ covered in our experiments just quoted. In the literature the effect of slowly decaying and different initial conditions in different experiments may have been confused with the effect of Reynolds number on the structure of these vortices; see, for example,

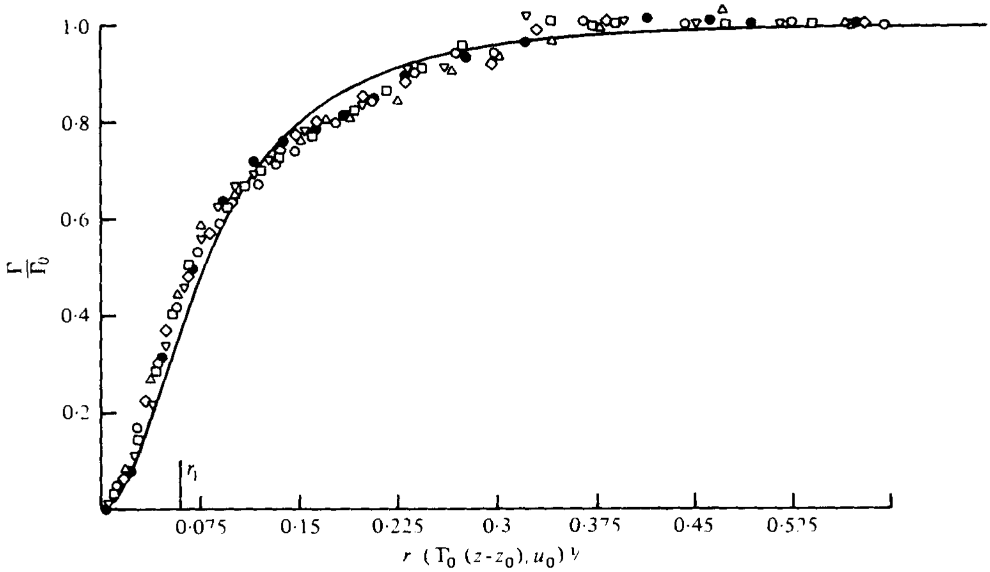

FIGURE 2. Comparison of the theory with experiments. ——, $a = 150$, $b = 10$; ●, $z/c = 40$; △, $z/c = 50$; ◇, $z/c = 60$; ○, $z/c = 70$; □, $z/c = 80$; ▽, $z/c = 85$. u_θ is a maximum at $r = r_1$.

McCormick *et al.* (1968). The terms involving ν may be neglected in (29)–(32) (Uberoi 1977*b*); thus

$$\Gamma/\Gamma_0 = \gamma(ru_0/\Gamma_0, zu_0/\Gamma_0) \tag{33}$$

and

$$u_0 \frac{\partial}{\partial z} u_\theta^2 r^2 = \frac{\partial}{\partial r} r^2(-u_r u_\theta - \overline{u_r' u_\theta'}) - \frac{\partial}{\partial z} u_z u_\theta r^2. \tag{34}$$

In order to proceed further we could write down the equations governing u_r and u_z and assume enough relations among the independent variables so that their number equals the number of equations. Instead we propose an elemental theory which incorporates the mechanism of vortex changes discussed above.

We look for a solution such that Γ/Γ_0 is a function of the single variable

$$\eta = \left(\frac{ru_0}{\Gamma_0}\right)^2 \left(\frac{\Gamma_0}{zu_0}\right)^n. \tag{35}$$

We assume that the total radial and associated axial convection of angular momentum is given by

$$\int_0^\infty \left[\frac{\partial}{\partial r} r^2(u_r u_\theta + \overline{u_r' u_\theta'}) + \frac{\partial}{\partial z} r^2 u_z u_\theta\right] dr \sim \left(\frac{\Gamma_0}{zu_0}\right)^m \Gamma_0^2. \tag{36}$$

The terms on the right-hand side of (34) are significant only in the core, where $\Gamma - \Gamma_0$ is significantly different from zero. The sign of these terms should not depend on the sign of $\Gamma - \Gamma_0$ and they should have proper dependence on r as $r \to 0$. On these bases and in the light of the discussion of the physical phenomena we propose that

$$\frac{\partial}{\partial r} r^2(u_r u_\theta + \overline{u_r' u_\theta'}) + \frac{\partial}{\partial z} r^2 u_z u_\theta = na \left(\frac{\Gamma_0}{zu_0}\right)^m (\Gamma - \Gamma_0)^2 \frac{\eta^2}{r} \exp(b\eta), \tag{37}$$

where a and b are constants. The factor $\exp b\eta$ recognizes the fact that the turbulent core at any z is of finite size and (37) should rapidly approach zero as we go from the turbulent core to the surrounding non-turbulent fluid. Using (37), the governing equation (34) becomes

$$u_0 \frac{\partial}{\partial z} \Gamma r = -na \left(\frac{\Gamma_0}{zu_0}\right)^m (\Gamma - \Gamma_0)^2 \frac{\eta^2}{r} \exp(b\eta) \tag{38}$$

and for $m = 1-n$ we have

$$d\gamma/d\eta = a(1-\gamma)^2 \exp(b\eta), \quad \gamma = \Gamma/\Gamma_0. \tag{39}$$

The solution is

$$\gamma = 1 - \left[1 + \frac{a}{b}(\exp(b\eta) - 1)\right]^{-1}. \tag{40}$$

This expression is compared with observations (Singh 1974; Uberoi 1974) in figure 2, where $n = 1$, $a = 150$ and $b = 10$. A very brief outline of the derivation of the above equation was published earlier (Uberoi 1977*c*).

6. Final stages in vortex decay

If $m > 0$ in (36) then relative to diffusion the importance of convection of angular momentum decreases as $z \to \infty$. Independent of the theory proposed here, let us assume that the axial velocity difference $u_z \ll u_0$ in the sense that the convection is negligible compared with the diffusion and the core size continues to increase at least at the rate given by the diffusion as $z \to \infty$. We claim that under these conditions the flow becomes stable and no sustained turbulence is possible. See § 2 above.

In studies of laminar and turbulent line vortices self-similarity is often assumed (Squire 1965; Govindaraju & Saffman 1971). Consider the following form:

$$\Gamma/\Gamma_0 = \gamma(r^2/ct^n) = \gamma(\xi). \tag{41}$$

Integrating (10) and using (41), we have

$$2\pi \int_0^\infty \frac{\partial}{\partial t} u_\theta r^2 dr = \frac{-n\Gamma_0 t^{n-1} c}{2} \int_0^\infty \gamma' \xi d\xi = -2\nu\Gamma_0. \tag{42}$$

It follows that $n = 1$ and $c \sim \nu$. Using (41), the expression for the rate of change of the kinetic energy of the mean motion becomes

$$\pi \int_0^\infty \frac{\partial}{\partial t} u_\theta^2 r dr = -\Gamma_0^2/8\pi t. \tag{43}$$

It may be shown that $\Gamma_0^2/8\pi t$ is the rate of viscous dissipation of kinetic energy of a laminar line vortex given by (7). In a line vortex with a turbulent core, the rate of decrease of kinetic energy should *exceed* that in a laminar line vortex with the same overall parameters. Hence, even if a turbulent line vortex exists, it cannot have the form given by (41). We may say that the virtual origins of t are different for laminar and turbulent line vortices. However, this becomes unimportant as $t \to \infty$.

The beginning of the final period of vortex and its dependence on initial conditions and the Reynolds number need further experimental study.

7. Discussion

We have shown that all previous theories of the decay of trailing vortices are based on assumptions which are untenable under most conditions of interest. We have found the dominant physical phenomena of radial and associated axial convection of angular momentum and the role of the axial velocity in sustaining turbulence in the vortex core.

A theory for a turbulent trailing vortex was presented which satisfies the requirement of turbulent theories. We proposed an expression for the dominant physical phenomena which is consistent with the basic equations, and the results agree with observations.

A reasonably complete discussion was presented because of confusion in this field and to provide suggestions for further experimental work in turbulent vortices. It is necessary to conduct more extensive experiments to determine accurately the values of m and n. Once we have accurate measurements of u_z we may use the following relation [obtained from (37)] to determine m independently:

$$\int_0^\infty \frac{\partial}{\partial z} u_z(\Gamma - \Gamma_0)\, r dr = na \left(\frac{\Gamma_0}{zu_0}\right)^m \int_0^\infty \left(\frac{\Gamma - \Gamma_0}{\Gamma_0}\right)^2 \eta \exp(b\eta)\, d\eta. \tag{44}$$

In the past there have been no guiding physical processes or theories which could help to evaluate various devices and methods for amelioration of the vortex-wake problem and its influence on the operation of aeroplanes, which may interact with trailing vortices from other aeroplanes. It is hoped that the present theory and discussion of physical phenomena will provide such guidance.

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